

A Note on \aleph_0 -injective Rings

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Abstract: A ring R is called right \aleph_0 -injective if every right homomorphism from a countably generated right ideal of R to R_R can be extended to a homomorphism from R_R to R_R . In this note, some characterizations of \aleph_0 -injective rings are given. It is proved that if R is semiperfect, then R is right \aleph_0 -injective if and only if every homomorphism from a countably generated small right ideal of R to R_R can be extended to one from R_R to R_R . It is also shown that if R is right noetherian and left \aleph_0 -injective, then R is QF . This result can be looked as an approach to the Faith-Menal conjecture.

Key Words: \aleph_0 -injective rings; Faith-Menal conjecture; Quasi-Frobenius rings.

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1. INTRODUCTION

Throughout this paper rings are associative with identity. Write J and S_l for the Jacobson radical and the left socle of a ring R respectively. Use $N \subseteq^{ess} M$ to mean that N is an essential submodule of M . For a subset X of a ring R , the left annihilator of X in R is $\mathbf{l}(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. Right annihilators are defined analogously. $f = c \cdot$ means that f is a homomorphism multiplied by an element c on the left side.

It is mentioned in [13] that a ring R is called right \aleph_0 -(or countably) *injective* if every right homomorphism from a countably generated right ideal of R to R_R can be extended to a homomorphism from R_R to R_R . Recall that a ring R is called right *F-injective* if every right homomorphism from a finitely generated

right ideal of R to R_R can be extended to one from R_R to R_R . And a right *FP-injective* ring R satisfies that every right homomorphism from a finitely generated submodule of a free right R -module F_R to R_R can be extended to one from F_R to R_R . The left side of the above injectivities can be defined similarly. It is obvious that right self-injective rings are right \aleph_0 -injective and right \aleph_0 -injective rings are right F -injective. But neither of the converses is true (see [13, Example 10.46]). The example also shows that a right *FP*-injective ring may not be right \aleph_0 -injective. But it is still unknown whether a right F -injective ring is right *FP*-injective. We have the following arrow diagrams on injectivities of rings:

$$\begin{array}{ccccccc} \text{right self-injectivity} & \xrightarrow{\neq} & \text{right } \aleph_0\text{-injectivity} & \xrightarrow{\neq} & \text{right } F\text{-injectivity} & \overset{?}{\rightleftarrows} & \text{right } FP\text{-injectivity,} \\ & & & & & & \\ & & \text{right self-injectivity} & \xrightarrow{\neq} & \text{right } FP\text{-injectivity} & \overset{?}{\rightleftarrows} & \text{right } \aleph_0\text{-injectivity.} \end{array}$$

Recall that a ring R is *quasi-Frobenius* (QF) if R is one-sided noetherian and one-sided self-injective. There are three unresolved Faith conjectures on QF rings (see [10]). One of them is the Faith-Menal conjecture, which was raised by Faith and Menal in [3]. The conjecture says that every strongly right Johns ring is QF . A ring R is called *right Johns* if R is right noetherian and every right ideal of R is a right annihilator. R is called *strongly right Johns* if the matrix ring $M_n(R)$ is right Johns for all $n \geq 1$. In [4], Johns used a false result of Kurshan [6, Theorem 3.3] to show that right Johns rings are right artinian. Later in [2], Faith and Menal gave a counter example to show that right Johns rings may not be right artinian. They characterized strongly right Johns rings as right noetherian and left *FP*-injective rings (see [3, Theorem 1.1]). So the Faith-Menal conjecture is equivalent to say that every right noetherian and left *FP*-injective ring is QF . In this short article, some characterizations of \aleph_0 -injective rings are explored. It is proved in Theorem 9 that if R is semiperfect, then R is right \aleph_0 -injective if and only if every homomorphism from a countably generated small right ideal of R to R_R can be extended to one from R_R to R_R . Since *FP*-injectivity implies F -injectivity, it is unknown whether a right noetherian and left F -injective ring is QF . It is proved in Theorem 14 that a right noetherian and left \aleph_0 -injective ring is QF . This result can be looked as an approach to the Faith-Menal conjecture.

2. RESULTS

First we explore some basic characterizations of \aleph_0 -injective rings.

Proposition 1. *A direct product of rings $R = \prod_{i \in I} R_i$ is right \aleph_0 -injective if and only if R_i is right \aleph_0 -injective, $\forall i \in I$.*

Proof. For $i \in I$, let π_i and ι_i be the i th projection map and the i th inclusion map canonically. If R is right \aleph_0 -injective, for each i , suppose $f_i : T_i \rightarrow R_i$ is R_i -linear where T_i is a countably generated right ideal of R_i . Then the map $0 \times \cdots \times T_i \times \cdots \times 0 \rightarrow 0 \times \cdots \times R_i \times \cdots \times 0$ given by $(0, \dots, t_i, \dots, 0) \mapsto (0, \dots, f_i(t_i), \dots, 0)$ is R -linear with $0 \times \cdots \times T_i \times \cdots \times 0$ a countably generated right ideal of R . So it has the form $c \cdot$ where $c \in R$. Thus $f_i = \pi_i(c) \cdot$. Conversely, let $\gamma : T \rightarrow R$ be R -linear, where T is a countably generated right ideal of R . For each $i \in I$, let $T_i = \{x \in R_i \mid \iota_i(x) \in T\}$. Since T is countably generated, $T = \sum_{k=1}^{\infty} a_k R$, where $a_k \in R$, $k = 1, 2, \dots$. Then it is easy to prove that $T_i = \sum_{k=1}^{\infty} \pi_i(a_k) R_i$ is a countably generated right ideal of R_i , $\forall i \in I$. Now define $\gamma_i : T_i \rightarrow R_i$ by $\gamma_i(x) = \pi_i \gamma(\iota_i(x))$, $x \in T_i$. Since R_i is right \aleph_0 -injective, there exists $c_i \in R_i$ such that $\gamma_i = c_i \cdot$. For each $\bar{t} = \langle t_i \rangle \in T$, write $\gamma(\bar{t}) = \bar{s} = \langle s_i \rangle$. Since T is a right ideal of R , $t_i \in T_i, \forall i \in I$. Thus $s_i = \pi_i(\bar{s} \cdot \iota_i(1_i)) = \pi_i(\gamma(\bar{t}) \cdot \iota_i(1_i)) = \pi_i \gamma(\bar{t} \cdot \iota_i(1_i)) = \pi_i \gamma(\iota_i(t_i)) = \gamma_i(t_i) = c_i t_i$, whence $\bar{s} = \langle c_i \rangle \cdot \bar{t}$. So $\gamma = \langle c_i \rangle \cdot$. This shows that R is right \aleph_0 -injective. \square

Proposition 2. *If R is right \aleph_0 -injective, then $\mathbf{l}(I \cap K) = \mathbf{l}(I) + \mathbf{l}(K)$, where I and K are countably generated right ideals of R .*

Proof. It is only to be shown that $\mathbf{l}(I \cap K) \subseteq \mathbf{l}(I) + \mathbf{l}(K)$. Let $x \in \mathbf{l}(I \cap K)$. Define a right R -homomorphism f from $I + K$ to R_R such that $f(i + k) = xi$, where $i \in I$ and $k \in K$. Then it is clear that f is well-defined. Since I and K are both countably generated right ideals of R , $I + K$ is also a countably generated right ideal of R . As R is right \aleph_0 -injective, f can be extended to a homomorphism from R_R to R_R . Hence there exist an element $c \in R$ such that $f = c \cdot$. Thus, by the definition of f , $c \in \mathbf{l}(K)$ and $x - c \in \mathbf{l}(I)$. So $x = (x - c) + c \in \mathbf{l}(I) + \mathbf{l}(K)$. \square

Recall that a ring R is called *right Kasch* if each simple right R -module can embed into R_R . Or equivalently, every maximal right ideal of R is a right annihilator. Left Kasch rings can be defined similarly.

Proposition 3. *If R is right Kasch and right \aleph_0 -injective, then every countably generated right ideals of R is a right annihilator.*

Proof. Let I be a countably generated right ideal of R . If I is not a right annihilator, then there exists a nonzero element $x \in R$ such that $x \in \mathbf{rl}(I) \setminus I$. Now let $K = I + xR$. Then $\overline{K} = K/I$ is finitely generated. Hence \overline{K} has a maximal submodule \overline{M} . Since R is right Kasch, $\overline{K}/\overline{M}$ can embed into R_R . Thus there exists a homomorphism f from K to R_R with $f(I) = 0$ and $f(x) \neq 0$. Since R is right \aleph_0 -injective, $f = c \cdot$ for some $c \in R$. So $c \in \mathbf{l}(I)$. Since $x \in \mathbf{rl}(I)$, $f(x) = cx = 0$. This is a contradiction. \square

Theorem 4. *Let R be a right \aleph_0 -injective ring. For any idempotent $e \in R$ with $ReR = R$, the corner ring eRe is also right \aleph_0 -injective.*

Proof. Let $S = eRe$ and $\theta : T \rightarrow S$ be a right S -homomorphism from a countably generated right ideal T of S to S_S . Define $\bar{\theta} : TR \rightarrow R_R$ by $\bar{\theta}(\sum t_i r_i) = \sum \theta(t_i) r_i$, $t_i \in T$, $r_i \in R$. Assume $\sum t_i r_i = 0$. For any $r \in R$, $0 = \sum t_i r_i r e = \sum t_i (e r_i r e)$. So $0 = \sum \theta(t_i) (e r_i r e) = [\sum \theta(t_i) r_i] r e$. Since $ReR = R$, it is clear that $\sum \theta(t_i) r_i = 0$. Hence $\bar{\theta}$ is a well-defined right R -homomorphism. Since T is a countably generated right ideal of S , TR is also a countably generated right ideal of R . As R is right \aleph_0 -injective, $\bar{\theta} = c \cdot$ for some $c \in R$. Then for each $t \in T$, $\theta(t) = e\theta(t) = e\bar{\theta}(t) = ect = (ec)et = (ece)t$. Hence $\theta = (ece) \cdot$, as required. \square

Remark 5. The condition that $ReR = R$ in the above theorem is necessary. For example (see [5, Example 9]), let R be the algebra of matrices over a field

$$K \text{ of the form } R = \begin{bmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}, \quad a, b, c, x, y, z \in K.$$

Set $e = e_{11} + e_{22} + e_{44} + e_{55}$, which is a sum of canonical matrix units. It is

clear that e is an idempotent of R such that $ReR \neq R$. R is right \aleph_0 -injective, but eRe is not right \aleph_0 -injective.

Proof. [5, Example 9] shows that R is a QF ring and eRe is not a QF ring. Since R is QF , R is right self-injective and eRe is left noetherian. So R is right \aleph_0 -injective. If eRe is right \aleph_0 -injective, then eRe is QF by Theorem 14. This is a contradiction. \square

It is natural to ask whether right \aleph_0 -injectivity is a Morita invariant.

Question 6. *If R is right \aleph_0 -injective, is $M_{n \times n}(R)$ ($n \geq 2$) right \aleph_0 -injective?*

The method in the proof of the following theorem is owing to [8, Theorem 1]

Theorem 7. *The following are equivalent for a ring R and an integer $n \geq 1$:*

- (1) $M_n(R)$ is right \aleph_0 -injective.
- (2) For each countably generated right R -submodule T of R_n , every R -linear map $\gamma: T \rightarrow R$ can be extended to $R_n \rightarrow R$.
- (3) For each countably generated right R -submodule T of R_n , every R -linear map $\gamma: T \rightarrow R_n$ can be extended to $R_n \rightarrow R_n$.

Proof. We prove for the case $n=2$. The others are analogous.

(1) \Rightarrow (2).

Given $\gamma: T \rightarrow R$ where T is a countably generated right R -submodule of R_2 , consider the countably generated right ideal $\overline{T} = [T \ T] = \{[\alpha \ \beta] | \alpha, \beta \in T\}$ of $M_2(R)$. The map $\overline{\gamma}: \overline{T} \rightarrow M_2(R)$ defined by

$$\overline{\gamma}([\alpha \ \beta]) = \begin{bmatrix} \gamma(\alpha) & \gamma(\beta) \\ 0 & 0 \end{bmatrix}, \alpha, \beta \in T$$

is $M_2(R)$ -linear. By (1), there exists $C \in M_2(R)$ such that $\overline{\gamma} = C \cdot$. So $\gamma = \alpha \cdot$, where α is the first row of C . Hence γ can be extended to a homomorphism from R_2 to R .

(2) \Rightarrow (3).

Given (2), consider $\gamma: T \rightarrow R_2$ where T is a countably generated right R -submodule of R_2 . Let $\pi_i: R_2 \rightarrow R$ be the i th projection, $i = 1, 2$. Then (2) provides an R -linear map $\gamma_i: R_2 \rightarrow R$ extending $\pi_i \circ \gamma$, $i = 1, 2$. Thus $\overline{\gamma}: R_2 \rightarrow R_2$ extends γ where $\overline{\gamma}(\overline{x}) = [\gamma_1(\overline{x}) \ \gamma_2(\overline{x})]^T$, $\overline{x} \in R_2$.

(3) \Rightarrow (1).

Write $S = M_2(R)$, consider $\gamma: T \rightarrow S_S$ where T is a countably generated right ideal of S . Then it is easy to prove that $T = [T_0 \ T_0]$ where $T_0 = \{\bar{x} \in R_2 \mid [\bar{x} \ 0] \in T\}$ is a right countably generated right R -submodule of R_2 . For $\bar{x} \in T_0$, the S -linearity of γ shows that $\gamma[\bar{x} \ 0] = \gamma([\bar{x} \ 0] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \gamma([\bar{x} \ 0]) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = [\bar{y} \ 0]$ for some $\bar{y} \in R_2$. Writing $\bar{y} = \gamma_0(\bar{x})$ yields an R -linear map $\gamma_0: T_0 \rightarrow R_2$ such that $\gamma[\bar{x} \ 0] = [\gamma_0(\bar{x}) \ 0]$, $\bar{x} \in T_0$. Then γ_0 extends to an R -linear map $\bar{\gamma}: R_2 \rightarrow R_2$ by (3). Hence $\gamma_0 = C \cdot$ for some $C \in S$. If $[\bar{x} \ \bar{y}] \in T$ it follows that $\gamma([\bar{x} \ \bar{y}]) = \gamma([\bar{x} \ 0] + [\bar{y} \ 0] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = [\gamma_0(\bar{x}) \ 0] + [\gamma_0(\bar{y}) \ 0] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = [C\bar{x} \ C\bar{y}] = C[\bar{x} \ \bar{y}]$. This shows $\gamma = C \cdot$. \square

Recall that a right ideal L of a ring R is called *small* if, for any proper right ideal L' of R , $L + L' \neq R_R$. Let I be a right ideal of R . I is said to *lie over* a direct summand of R_R if there exists an idempotent $e \in R$ such that $I = eR \oplus (I \cap (1 - e)R)$, where $I \cap (1 - e)R$ is a small right ideal of R .

Lemma 8. [7, Corollary 2.10] *A ring R is semiperfect if and only if every countably generated right ideal of R lies over a direct summand of R_R .*

Theorem 9. *Let R be semiperfect. If every homomorphism from a countably generated small right ideal of R to R_R can be extended to one from R_R to R_R , then R is right \aleph_0 -injective.*

Proof. Let I be a countably generated right ideal of R and f be a homomorphism from I to R_R . By the above lemma, $I = eR \oplus K$, where e is an idempotent of R_R and $K = I \cap (1 - e)R$ is a small right ideal of R . Since I is countably generated, K is also countably generated. By hypothesis, there exists a homomorphism g from $(1 - e)R$ to R_R such that $g|_K = f|_K$. For each $x \in R$, define $F(x) = f(x_1) + g(x_2)$ where $x_1 = ex$ and $x_2 = (1 - e)x$. It is clear that $F|_I = f$. \square

Now we turn to the main theorem of this note. First look at some lemmas.

Lemma 10. *If R is a left \aleph_0 -injective ring with ACC on right annihilators, then R is left finite dimensional.*

Proof. Assume R is not left finite dimensional. Then there are nonzero elements $a_i \in R, i = 1, 2, \dots$, such that $\{Ra_i\}_{i=1}^\infty$ is an independent family of proper left ideals of R . Let $I_k = \bigoplus_{i=k}^\infty Ra_i, k = 1, 2, \dots$. Then $\mathbf{r}(I_1) \subseteq \mathbf{r}(I_2) \subseteq \dots$. Since R satisfies ACC on right annihilators, there exists $n \in \mathbb{N}$ such that $\mathbf{r}(I_n) = \mathbf{r}(I_{n+1})$. As $I_n = I_{n+1} \oplus Ra_n$, we have $\mathbf{r}(I_n) = \mathbf{r}(I_{n+1}) \cap \mathbf{r}(a_n)$. So $\mathbf{r}(I_n) \subseteq \mathbf{r}(a_n)$. Since R is left \aleph_0 -injective, by the symmetry of Proposition 2, $R = \mathbf{r}(0) = \mathbf{r}(I_{n+1} \cap Ra_n) = \mathbf{r}(I_{n+1}) + \mathbf{r}(a_n) = \mathbf{r}(I_n) + \mathbf{r}(a_n) = \mathbf{r}(a_n)$. Thus $a_n = 0$. This is a contradiction. \square

Recall that a ring R is called left P -injective (2-injective) if every homomorphism from a principal (2-generated) left ideal of R to ${}_R R$ can be extended to one from ${}_R R$ to ${}_R R$.

Lemma 11. [9, Theorem 3.3] *If R is left P -injective and left finite dimensional, then R is semilocal.*

Lemma 12. [1, Theorem 2.7] *If R is right noetherian and left P -injective, then J is nilpotent.*

Lemma 13. [11, Corollary 3] *If R is a left 2-injective ring with ACC on left annihilators, then R is QF.*

Now we obtain the main theorem.

Theorem 14. *If R is right noetherian and left \aleph_0 -injective, then R is QF.*

Proof. Since R is right noetherian, R satisfies ACC on right annihilators. By Lemma 10, R is left finite dimensional. Since R is left \aleph_0 -injective, R is left P -injective. So R is semilocal and J is nilpotent by Lemma 11 and Lemma 12. Thus R is semiprimary. Hence R is right artinian. So R satisfies ACC on left annihilators. Then R is QF by Lemma 13. \square

By Lemma 13, we see that if R is a left \aleph_0 -injective ring with ACC on left annihilators, then R is QF. It is natural to ask the following question:

Question 15. *Can right noetherian condition in the above theorem be weakened to the condition satisfying ACC on right annihilators?*

Remark 16. The answer is "yes" if we can show that J is right T -nilpotent. By Lemma 10 and Lemma 11, R is semilocal. If J is right T -nilpotent, then

R is right perfect. So R is left GPF (i.e., R is left P -injective, semiperfect and $S_l \subseteq^{ess} R$). Thus R is left Kasch by [10, Theorem 5.31]. By [9, Lemma 2.2], R is right P -injective. So R is left and right mininjective. Recall that a ring R is called right *mininjective* if every homomorphism from a minimal right ideal of R to R_R can be extended to one from R_R to R_R . Left mininjective rings can be defined similarly. Then by [12, Theorem 2.5], R is QF .

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